TRIANGLES IN A PLANAR MEASURABLE SET

Lemma 1. *Let $p_1, p_2, p_3, p'_1, p'_2, p'_3 \in \mathbb{R}^2$. If $\|p_i - p'_i\| \le \varepsilon$ $(i = 1, 2, 3)$, then we have*

$$\bigl|\mu(\triangle p_1p_2p_3) - \mu(\triangle p'_1p'_2p'_3)\bigr| \le 2\varepsilon \operatorname{diam}(\triangle p_1p_2p_3) + 2\varepsilon^2.$$

Proof. Let $u_i = p'_i - p_i$ and $d = \operatorname{diam}(\triangle p_1p_2p_3)$. Then we have

$$\begin{aligned}
\mu(\triangle p_1p_2p_3) &= \frac{1}{2}\bigl|\det(p_2 - p_1, p_3 - p_1)\bigr|, \\
\mu(\triangle p'_1p'_2p'_3) &= \frac{1}{2}\bigl|\det(p'_2 - p'_1, p'_3 - p'_1)\bigr| \\
&= \frac{1}{2}\bigl|\det(p_2 - p_1, p_3 - p_1) + \det(u_2 - u_1, p_3 - p_1) \\
&\qquad + \det(p_2 - p_1, u_3 - u_1) + \det(u_2 - u_1, u_3 - u_1)\bigr|.
\end{aligned}$$

Using $\bigl|\det(a, b)\bigr| \le \|a\| \cdot \|b\|$, we get

$$\begin{aligned}
&\bigl|\mu(\triangle p_1p_2p_3) - \mu(\triangle p'_1p'_2p'_3)\bigr| \\
&\le \frac{1}{2}\bigl(\|u_2 - u_1\| \cdot \|p_3 - p_1\| + \|p_2 - p_1\| \cdot \|u_3 - u_1\| + \|u_2 - u_1\| \cdot \|u_3 - u_1\|\bigr) \\
&\le \frac{1}{2}\bigl(2\varepsilon \cdot d + d \cdot 2\varepsilon + (2\varepsilon)^2\bigr) \\
&= 2\varepsilon d + 2\varepsilon^2.
\end{aligned}$$

This completes the proof. $\square$

Let $N > 0$. We say that a set of lattice points $S \subset \mathbb{Z}^2$ is *N-avoiding* if for any $q_1, q_2, q_3 \in S$, we have

$$\bigl|\mu(\triangle q_1q_2q_3) - N\bigr| > \operatorname{diam}(\triangle q_1q_2q_3).$$

Taking (q, q, q') with $q, q' \in S$, we see that the diameter of an N-avoiding set is at most N. In particular, the largest possible size $f(N)$ of an N-avoiding set exists.

Theorem 2. *The following assertions are equivalent:*

1. *There exists a constant $c > 0$ such that, for every measurable set $A \subset \mathbb{R}^2$ with $\mu(A) > c$, there exist three points $p_1, p_2, p_3 \in A$ that form a triangle of area 1.*
2. $f(N) = O(N)$.

Proof. We first prove (1) $\implies$ (2). Take a constant $c > 0$ as in (1). Let $N > 0$ and let $S \subset \mathbb{Z}^2$ be an N-avoiding set. For each $q \in S$, let $D(q)$ be the closed disk of radius $1/(3\sqrt{N})$ centered at $q/\sqrt{N}$. Define

$$D = \bigcup_{q \in S} D(q).$$

We will show that there do not exist $p_1, p_2, p_3 \in D$ that form a triangle of area 1. Suppose that $p_1, p_2, p_3 \in D$ form a triangle of area 1. Let q_i be the unique element of S satisfying $p_i \in D(q_i)$. Since $\mu(D(q_i)) = \pi/(9N) < 1$, it cannot happen that $q_1 = q_2 = q_3$. If we set $q'_i = \sqrt{N} \cdot p_i$, then we have $\mu(\triangle q'_1q'_2q'_3) = N$. Since $p_i \in C(q_i)$, we have $\|q_i - q'_i\| \le 1/3$ and hence

$$\bigl|\mu(\triangle q_1q_2q_3) - N\bigr| \le \frac{2}{3}\operatorname{diam}(\triangle q_1q_2q_3) + \frac{2}{9}$$

by Lemma 1. Moreover, we have $\mathrm{diam}(\triangle q_1q_2q_3) \geq 1$ since $q_i \in \mathbb{Z}^2$. Therefore we have

$$\left|\mu(\triangle q_1q_2q_3) - N\right| \leq \mathrm{diam}(\triangle q_1q_2q_3).$$

This contradicts the assumption that S is N-avoiding. By assumption (1), we get

$$\mu(D) = \#S \cdot \frac{\pi}{9N} \leq c$$

and hence $\#S \leq (9c/\pi) \cdot N$. This proves $f(N) = O(N)$.

Next we prove (2) $\Longrightarrow$ (1). Let $C > 0$ be a constant such that $f(N) \leq CN$ holds for $N > 0$. Let $A \subset \mathbb{R}^2$ be a measurable set such that $\mu(A) > 4\pi C$. We will show that there exist three points $p_1, p_2, p_3 \in A$ that form a triangle of area 1. By the inner regularity of the Lebesgue measure, we may assume that A is compact. For $N > 0$ and $q \in \mathbb{Z}^2$, we write $E_N(q)$ for the closed disk with radius $2/\sqrt{N}$ centered at $q/\sqrt{N}$. We define S_N to be the set of lattice points $q \in \mathbb{Z}^2$ such that $E_N(q) \cap A \neq \emptyset$. Since $\{E_N(q) \mid q \in \mathbb{Z}^2\}$ covers $\mathbb{R}^2$, we have

$$A \subset \bigcup_{q \in S_N} E_N(q) =: E_N.$$

Moreover, we have

$$E_N \subset \overline{\bigcup_{a \in A} B\left(a, \frac{4}{\sqrt{N}}\right)}$$

and hence $A = \bigcap_{N>0} E_N$. Therefore it suffices to show that for each $N > 0$, there exist three points $p_1, p_2, p_3 \in E_N$ that form a triangle of area 1. We have

$$\#S_N \cdot \frac{4\pi}{N} \geq \mu(E_N) \geq \mu(A) > 4\pi C$$

and hence $\#S_N > CN \geq f(N)$. Therefore, there exist $q_1, q_2, q_3 \in S_N$ such that

$$\left|\mu(\triangle q_1q_2q_3) - N\right| \leq \mathrm{diam}(\triangle q_1q_2q_3).$$

Let $p_i = q_i/\sqrt{N}$. Then we have

$$\left|\mu(\triangle p_1p_2p_3) - 1\right| \leq \frac{1}{\sqrt{N}} \cdot \mathrm{diam}(\triangle p_1p_2p_3).$$

Without loss of generality, we may assume that $\mathrm{diam}(\triangle p_1p_2p_3) = \|p_1 - p_2\|$. As p_3' varies within $E_N(q_3)$, the area $\mu(\triangle p_1p_2p_3')$ ranges continuously over the entire interval

$$\max\left\{0, \mu(\triangle p_1p_2p_3) - \frac{\|p_1 - p_2\|}{\sqrt{N}}\right\} \leq \mu(\triangle p_1p_2p_3') \leq \mu(\triangle p_1p_2p_3) + \frac{\|p_1 - p_2\|}{\sqrt{N}}.$$

In particular, there exists some $p_3' \in E_N(q_3)$ such that $\mu(\triangle p_1p_2p_3') = 1$. This completes the proof. $\square$