COHOMOLOGY OF QUASI-COHERENT SHEAVES ON AFFINE SCHEMES

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In this note we present two proofs of the following theorem.

Theorem 0.1. *Let X be an affine scheme and F a quasi-coherent sheaf on X. Then we have $H^n(X,F)=0$ for $n>0$.*

We write $F_U := j_*(F|_U)$ where $j\colon U \to X$ denotes the inclusion.

1. Proof via Leray spectral sequence

First we show that $H^1(X,F)=0$. Suppose that there is an exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{QCoh}(X)$. Since $\Gamma(X,-)\colon \mathbf{QCoh}(X) \to \mathbf{Ab}$ is an exact functor, $\Gamma(X,G) \to \Gamma(X,H)$ is surjective and hence

$$H^1(X,F) \to H^1(X,G)$$

is injective. Therefore it suffices to find an appropriate G for each $\alpha \in H^1(X,F)$ so that the image of α in $H^1(X,G)$ vanishes. We cannot use the Godement resolution since G has to be quasi-coherent, but we can imitate it inside $\mathbf{QCoh}(X)$ as follows.

Take an open covering $\{U_i\}_{i=1}^r$ of X by fundamental open subsets so that $\alpha|_{U_i}=0$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q j_{i*}(F|_{U_i})) \Rightarrow H^{p+q}(U, F|_{U_i}) \tag{1.1}$$

where $j_i\colon U_i \to X$ is the inclusion. We see that the canonical map $H^1(X,F_{U_i}) \to H^1(U_i,F|_{U_i})$ is injective. Hence the image of α in $H^1(X,\bigoplus_{i=1}^r F_{U_i})$ vanishes and $G := \bigoplus_{i=1}^r F_{U_i}$ will do the job.

Now we proceed to the case $n \geq 2$. Suppose that there is an exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{QCoh}(X)$. By the induction hypothesis, $H^{n-1}(X,H)=0$ and hence

$$H^n(X,F) \to H^n(X,G)$$

is injective. Therefore it suffices to find an appropriate G for each $\alpha \in H^n(X,F)$ so that the image of α in $H^n(X,G)$ vanishes. Again we will imitate the Godement resolution.

Take an open covering $\{U_i\}_{i=1}^r$ of X by fundamental open subsets so that $\alpha|_{U_i}=0$. Consider the Leray spectral sequence (1.1). By the induction hypothesis, we have $H^p(X,F_{U_i}) = 0$ for $0<p<n$ and $R^q j_*(F|_{U_i}) = 0$ for $0<q<n$. This implies that $E_2^{p,q}=0$ for $0<p+q<n$ and the canonical morphism $H^n(X,F_{U_i}) \to H^n(U_i,F|_{U_i})$ is injective. Hence the image of α in $H^n(X,\bigoplus_{i=1}^r F_{U_i})$ vanishes and $G := \bigoplus_{i=1}^r F_{U_i}$ will do the job.

2. Proof via Čech cohomology

First we show that $\check{H}^n(X, F) = 0$ for $n > 0$. It suffices to prove $\check{H}^n(\mathcal{U}, F) = 0$ for any open covering $\mathcal{U} = \{U_i\}_{i=1}^r$ of X by fundamental open subsets and $n > 0$. We may assume that we can write $X = \operatorname{Spec} A$, $U_i = D(f_i)$ and $F = \widetilde{M}$. Set $B = \prod_{i=1}^r A_{f_i}$. Then the Čech complex $\check{C}^\bullet(\mathcal{U}, F)$ is isomorphic to

$$M \otimes_A B \xrightarrow{d^0} M \otimes_A B \otimes_A B \xrightarrow{d^1} M \otimes_A B \otimes_A B \otimes_A B \xrightarrow{d^2} \cdots$$

where

$$d^k(m \otimes b_1 \otimes \cdots \otimes b_{k+1}) = \sum_{i=0}^{k+1} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_i \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_{k+1}.$$

It suffices to show that

$$C^\bullet = (0 \to M \xrightarrow{d^{-1}} M \otimes_A B \xrightarrow{d^0} M \otimes_A B \otimes_A B \xrightarrow{d^1} M \otimes_A B \otimes_A B \otimes_A B \xrightarrow{d^2} \cdots)$$

is exact where $d^{-1}(m) = m \otimes 1$. Since B is faithfully flat over A, it suffices to show the exactness of $C^\bullet \otimes_A B$. However, $C^\bullet \otimes_A B$ has a chain contraction $\{h^k \colon C^k \otimes_A B \to C^{k-1} \otimes_A B\}_{k=-1}^\infty$ given by

$$h_k((m \otimes b_1 \otimes \cdots \otimes b_{k+1}) \otimes b) = (m \otimes b_1 \otimes \cdots \otimes b_k) \otimes b_{k+1} b.$$

Now we deduce $H^n(X, F) = 0$ for $n > 0$ using the "Čech-to-derived functor" spectral sequence

$$E_2^{p,q} = \check{H}^n(X, \mathcal{H}^q(X, F)) \Rightarrow H^{p+q}(X, F)$$

where $\mathcal{H}^q(X, F)$ denotes the presheaf defined by $U \mapsto H^q(U, F)$. We use an induction on n. We have $E_2^{0,q} = 0$ for $q > 0$ since it injects to $\Gamma(X, \mathcal{H}^q(X, F)_{\mathrm{Zar}}) = 0$. We also have $E_2^{p,q} = 0$ for $0 < q < n$ by the induction hypothesis. Finally, we have $E_2^{p,0} = 0$ for $p > 0$ by what we proved above. These results show that $E_2^{p,q} = 0$ for $p + q = n$ and hence $H^n(X, F) = 0$.

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