BRAUER GROUPS

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Contents

1. Simple algebras

Fix a field K and its algebraic closure $\overline{K}$. A finite dimensional K-algebra A is called *simple* if it has exactly two two-sided ideals, namely 0 and A. Note that if A is simple, then any morphism from A to a non-zero K-algebra is injective. In particular, if $f\colon A \to B$ is a morphism of K-algebras with A simple and $[A : K] = [B : K]$, then f is an isomorphism. We say that A is a *division algebra* over K if $A^\times = A \setminus \{0\}$.

Example 1.1. Define $\mathbb{H}$ to be an $\mathbb{R}$-vector space with basis $1, i, j, k$ and define an $\mathbb{R}$-algebra structure on $\mathbb{H}$ by

$$i^2 = j^2 = -1, \quad ij = -ji = k.$$

Then $\mathbb{H}$ is a division algebra over $\mathbb{R}$ since

$$(a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2.$$

Example 1.2. Let D be a division algebra over K and $n \geq 1$. We prove that $M_n(D)$ is a simple K-algebra. For any $M = (m_{ij})_{i,j} \in M_n(D)$ with $m_{rs} \neq 0$, we have

$$E_{pr} M E_{sq} = m_{rs} E_{pq}$$

where E_{kl} denotes the matrix whose (k, l)-component is 1 and other components are 0. Since m_{rs} is invertible, this shows that the two-sided ideal generated by M is the whole $M_n(D)$.

Actually, any simple K-algebra is isomorphic to one given as in the above example.

Theorem 1.3 (Wedderburn)**.** *For any simple K-algebra A, there is a division algebra D over K and $n \geq 1$ such that $A \simeq M_n(D)$. Moreover, D is uniquely determinied up to isomorphism (we call D the division algebra associated to A).*

Lemma 1.4. *Let A be a simple K-algebra. Then there is a simple right A-module I such that any finitely generated right A-module is isomorphic to $I^{\oplus r}$ for some $r \geq 0$.*

Proof. Take a simple right A-submodule $I \subset A$. Since A is simple, we have $\sum_{a \in A} aI = A$ and hence there is a surjection $I^{\oplus r} \to A$ of right A-modules for some $r \geq 1$. Therefore, for any finitely generated right A-module M, there is a surjection $I^{\oplus N} \to M$ of right A-modules for some $N \geq 1$. Since $I^{\oplus N}$ is semisimple (i.e. finite direct sum of simple modules), so is M. If $M \simeq J_1 \oplus \cdots \oplus J_s$ with J_i a simple right A-module, then there is a non-trivial A-homomorphism $I \to J_i$ and hence $I \simeq J_i$. $\square$

Proof of Wedderburn's theorem. By Lemma 1.4 we have $A \simeq I^{\oplus n}$ as right A-modules for some $n \geq 1$. Then $A \simeq \mathrm{End}_{\mathbf{Mod}A}(A) \simeq \mathrm{End}_{\mathbf{Mod}A}(I^{\oplus n}) \simeq M_n(\mathrm{End}_{\mathbf{Mod}A}(I))$ and $D := \mathrm{End}_{\mathbf{Mod}A}(I)$ is a division algebra since I is simple. Since I in Lemma 1.4 is unique up to isomorphism, so is D. $\square$

Note that the class of simple algebras is not closed under tensor products over K. For example, if L is a Galois extension of K of degree n, then $L \otimes_K L \simeq L^n$ is not simple.

2. Central simple algebras

A K-algebra A is called *central* if its center $C(A)$ is equal to K. A standard argument shows $C(M_n(A)) = C(A)$ for any K-algebra A. For a K-algebra A and its K-subalgebra R, we define $C_A(R) = \{a \in A \mid \forall r \in R,\ ar = ra\}$.

Lemma 2.1. *Let A, B be K-algebras and $R \subset A$, $S \subset B$ be K-subalgebras. Then we have $C_{A \otimes_K B}(R \otimes_K S) = C_A(R) \otimes_K C_B(S)$. In particular, we have $C(A \otimes_K B) = C(A) \otimes_K C(B)$, hence if A and B are central then so is $A \otimes_K B$.*

Proof. The inclusion $C_{A \otimes_K B}(R \otimes_K S) \supset C_A(R) \otimes_K C_B(S)$ is clear. To prove the inverse inclusion, we choose a K-basis $\{e_\lambda\}_\lambda$ of B. Then any element $c = \sum_\lambda a_\lambda \otimes e_\lambda$ of $C_{A \otimes_K B}(R \otimes_K S)$ commutes with $r \otimes 1$ for $r \in R$, so we have $a_\lambda \in C_A(R)$ and hence

$$C_{A \otimes_K B}(R \otimes_K S) \subset C_A(R) \otimes_K B.$$

Similarly we have

$$C_{A \otimes_K B}(R \otimes_K S) \subset A \otimes_K C_B(S)$$

and these imply the desired inclusion. $\square$

Now we turn to the main subject of this note: central simple algebras (CSAs). Since $C(M_n(A)) = C(A)$, a simple K-algebra is central if and only if $A \simeq M_n(D)$ for some central division algebra (CDA) D. The next theorem shows that the class of CSA over K is closed under tensor products over K.

Theorem 2.2. *Let A, B be simple K-algebras. If B is central, then $A \otimes_K B$ is simple.*

Before proving this theorem, we need some preparation. Let D be a division algebra over K, V a free left D-module with basis $\{e_\lambda\}_\lambda$ and $W \subset V$ a left D-submodule. A non-zero element $w = \sum_\lambda a_\lambda e_\lambda \in W$ is called *primitive* (with respect to $\{e_\lambda\}_\lambda$) if $J(w) = \{\lambda \mid a_\lambda \neq 0\}$ is minimal among non-zero elements in W.

Lemma 2.3. *In the situation above, we have the following.*

(i) *If $w, w' \in W$ are primitive and $J(w) = J(w')$, then $w = cw'$ for some $c \in D^\times$.*
(ii) *As a left D-module, W is generated by primitive elements.*

Proof.

(i) Write $w = \sum_\lambda a_\lambda e_\lambda$ and $w' = \sum_\lambda b_\lambda e_\lambda$. Take $\lambda \in J(w)$ and consider the element $w - a_\lambda b_\lambda^{-1} w' \in W$. By the minimality of $J(w)$, we get $w - a_\lambda b_\lambda^{-1} w' = 0$.

(ii) For any non-zero element $w \in W$, we can choose a primitive element w' so that $J(w - w') \subsetneq J(w)$. Repeating this for $w - w'$, we can express w as a sum of primitive elements.

□

Lemma 2.4. *Let V be a K-vector space and D a CDA over K. Then any (D, D)-submodule of $D \otimes_K V$ is of the form $D \otimes_K V'$ for some K-subspace $V' \subset V$.*

Proof. Take a K-basis $\{e_\lambda\}_\lambda$ of V. Then $D \otimes_K V$ has a basis $\{1 \otimes e_\lambda\}_\lambda$ as a left D-module. Let W be a (D, D)-submodule of $D \otimes_K V$. By Lemma 2.3 (ii), it suffices to prove that any primitive element $w \in W$ is of the form $1 \otimes v$. Write $w = \sum_\lambda c_\lambda \otimes e_\lambda$. Then for any $d \in D$ we have

$$dw - wd = \sum_\lambda (dc_\lambda - c_\lambda d) \otimes e_\lambda \in W,$$

so the minimality of $J(w)$ implies that $c_\lambda \in C(D) = K$. □

Proof of Theorem 2.2. We have $B \simeq M_n(D)$ for some CDA D over K and $n \geq 1$. Then $A \otimes_K B \simeq M_n(A \otimes_K D)$, so it suffices to show that $A \otimes_K D$ is simple. Lemma 2.4 shows that any two-sided ideal of $A \otimes_K D$ is of the form $I \otimes_K D$ for some two-sided ideal $I \subset A$. Since A is simple, it follows that $A \otimes_K D$ is simple. □

Corollary 2.5 (Artin-Whaple)**.** *For any CSA A over K with $[A : K] = n$, the canonical homomorphism of K-algebras*

$$A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_{\mathbf{Vect}_K}(A) \simeq M_n(K); \quad a \otimes b \mapsto (x \mapsto axb)$$

is an isomorphism.

Proof. By Theorem 2.2, we see that $A \otimes_K A^{\mathrm{op}}$ is simple. The claim follows from this and $[A \otimes_K A^{\mathrm{op}} : K] = n^2 = [M_n(K) : K]$. □

Two CSAs A and B over K are called *similar* if the associated CDAs are isomorphic, and write $A \sim B$. In other words, we have $A \sim B$ if and only if

$$M_m(A) \simeq M_n(B)$$

holds for some $m, n \geq 1$. Let $\mathrm{Br}(K)$ denote the set of equivalence classes of CSAs over K with respect to this relation. By Theorem 2.2 and the Artin-Whaple theorem, we can define an abelian group structure on $\mathrm{Br}(K)$ by

$$[A] + [B] = [A \otimes_K B], \quad 0 = [K], \quad -[A] = [A^{\mathrm{op}}].$$

We call this group $\mathrm{Br}(K)$ the *Brauer group* of K.

Lemma 2.6. *Let A be a K-algebra and L/K a (possibly infinite dimensional) field extension. Then the following are equivalent:*

(i) *K is a CSA over K.*

(ii) *$L \otimes_K A$ is a CSA over L.*

Proof. First we prove (i) $\Longrightarrow$ (ii). We have $A \simeq M_n(D)$ for some CDA D over K and $n \geq 1$, so $L \otimes_K A \simeq M_n(L \otimes_K D)$. By Lemma 2.4, any two-sided ideal of $L \otimes_K D$ is of the form $L \otimes_K I$ for some two-sided ideal $I \subset D$. Since D is simple, it follows that $L \otimes_K D$ is simple and hence so is $L \otimes_K A$. Moreover, we have $C(L \otimes_K A) = C(L \otimes_K D) = C(L) \otimes_K C(A) = L$.

Next we prove (ii) $\Longrightarrow$ (i). Since $C(L \otimes_K A) = C(L) \otimes_K C(A) = L \otimes_K C(A)$, we have $C(A) = K$. If A is non-simple, then $L \otimes_K A$ is also non-simple, which is a contradiction. $\square$

Hence we can define a group homomorphism $R_{L/K} \colon \mathrm{Br}(K) \to \mathrm{Br}(L)$ by $[A] \mapsto [L \otimes_K A]$. We define $\mathrm{Br}(L/K)$ to be its kernel. We say that a CSA A over K *splits* over L if $[A] \in \mathrm{Br}(L/K)$, i.e. $L \otimes_K A$ is isomorphic to a matrix algebra over L.

Lemma 2.7. *Any K-subalgebra of a division algebra over K is again a division algebra.*

Proof. Let D be a division algebra over K and A its K-subalgebra. For any $a \in A \setminus \{0\}$, the K-linear map $A \to A$; $b \mapsto ab$ is injective. Since $[A : K] < \infty$, it is surjective and hence a is right invertible. A Similar argument shows that a is also left invertible. $\square$

In particular, for any $x \in D$, the K-subalgebra $K[x]$ of D generated by x is a field.

Theorem 2.8. *If K is algebraically closed, then* $\mathrm{Br}(K) = 0$.

Proof. Let D be a CDA over K. For any $x \in D$, $K[x]$ is a finite extension field of K and hence $K[x] = K$. This proves $D = K$ and hence the result. $\square$

Let A be a CSA over K. Then $\overline{K} \otimes_K A \simeq M_n(\overline{K})$ for some $n \geq 1$ by Theorem 2.8. It follows that $[A : K] = n^2$ is a perfect square. A limit argument shows that A splits over some finite extension L/K. Therefore we have

$$\mathrm{Br}(K) = \bigcup_{L \in \Phi_K} \mathrm{Br}(L/K)$$

where Φ_K is the set of finite extensions of K contained in $\overline{K}$.

3. Centralizer theorem

Theorem 3.1 (Skolem-Noether). *Let A, B be simple K-algebras and suppose that B is central. Then for any two K-homomorphisms $f, g \colon A \to B$ there is an inner automorphism $h \colon B \to B$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle h} \\ & & B. \end{array}$$

Proof. Via the isomorphism $B \simeq \mathrm{End}_{\mathbf{Mod}B}(B)$, we see that giving a K-homomorphism $f \colon A \to B$ is equivalent to giving an (A, B)-bimodule whose underlying right B-module is B. Moreover, an inner automorphism $B \to B$ corresponds to an automorphism of B as a right B-module. Therefore it suffices to show that two (A, B)-bimodules are isomorphic if their underlying right B-modules are the same. This follows from Lemma 1.4 applied to the simple K-algebra $A^{\mathrm{op}} \otimes_K B$. $\square$

Theorem 3.2 (Centralizer theorem)**.** *Let A be a CSA over K and B its simple K-subalgebra with $[B:K]=n$. Then the following hold:*

(i) *There is a non-canonical isomorphism $A \otimes_K B^{\mathrm{op}} \simeq M_n(C_A(B))$.*
(ii) $[A:K]=[B:K][C_A(B):K]$.
(iii) *$C_A(B)$ is simple.*
(iv) *$C_A(C_A(B))=B$ and $C(C_A(B))=C(B)$.*

Proof.

(i) $E := A \otimes_K \mathrm{End}_{\mathbf{Vect}_K}(B)$ has two isomorphic K-subalgebras $K \otimes_K B$ and $B \otimes_K K$. By Skolem-Noether theorem, there is an inner automorphism $h\colon E \to E$ such that the following diagram commutes:

$$\begin{array}{ccc} K \otimes_K B & \hookrightarrow & E \\ \downarrow \sim & & \downarrow h \\ B \otimes_K K & \hookrightarrow & E. \end{array}$$

Then h induces an isomorphism between $C_E(K \otimes_K B) = A \otimes B^{\mathrm{op}}$ and $C_E(B \otimes_K K) = C_A(B) \otimes_K \mathrm{End}_{\mathbf{Vect}_K}(B) \simeq M_n(C_A(B))$.

(ii) Taking $\dim_K$ in (i), we get $[A:K][B:K]=[B:K]^2[C_A(B):K]$.

(iii) If $C_A(B)$ is non-simple, then $M_n(C_A(B)) \simeq C_A(B) \otimes_K M_n(K)$ is also non-simple, which contradicts (i).

(iv) Clearly we have $B \subset C_A(C_A(B))$. However, we have $[C_A(C_A(B)):K] = \dfrac{[A:K]}{[C_A(B):K]} = [B:K]$ by (ii), so the equality holds. Finally, $C(C_A(B)) = C_A(B) \cap C_A(C_A(B)) = C_A(B) \cap B = C(B)$.

□

We use this theorem to study splittings of CSAs. Let A be a CSA over K and L/K a field extension contained in A. We say that L is a *special subfield* of A if $L = C_A(L)$ holds. In this case A splits over L since

$$L \otimes_K A \simeq M_n(C_A(L)) = M_n(L)$$

by (i) of the centralizer theorem, where $n=[L:K]$. The following theorem says that any splitting occurs essentially in this way.

Theorem 3.3. *Let A be a CSA over K and L/K a field extension contained in A. If A splits over L, then there is a CSA B over K with $A \sim B$ such that B contains a special subfield isomorphic to L. Moreover, such B is unique up to isomorphism.*

Lemma 3.4. *Let A be a CSA over K and L/K a field extension contained in A. Then the following are equivalent:*

(i) *L is a special subfield of A.*
(ii) $[A:K]=[L:K]^2$.

Proof. Since $L \subset C_A(L)$, the claim follows from (ii) of the centralizer theorem. □

Proof of Theorem 3.3. The uniqueness of B follows from $[B:K]=[L:K]^2$. Let us prove the existence. Set $[A:K]=n^2$ and $[L:K]=d$. Since A splits over L, we have

$$L \otimes_K A \simeq M_n(L) \subset \mathrm{End}_{\mathbf{Vect}_K}(L^{\oplus n}) =: R.$$

Let us prove that $B := C_R(A)^{\mathrm{op}}$ satisfies the required condition. First we see that B is a CSA by the centralizer theorem. We have $A^{\mathrm{op}} \sim B^{\mathrm{op}}$ because

$$R \otimes_K A^{\mathrm{op}} \simeq M_{n^2}(C_R(A))$$

by (i) of the centralizer theorem. Finally, L is a special subfield of B since

$$[B : K] = \frac{[R : K]}{[A : K]} = \frac{n^2 d^2}{n^2} = d^2$$

by (ii) of the centralizer theorem. □

4. Existence of a separable splitting field

Theorem 4.1. *Any CSA over K splits over some finite separable extension of K.*

In other words, we have

$$\mathrm{Br}(K) = \bigcup_{L \in \Lambda_K} \mathrm{Br}(L/K)$$

where Λ_K is the set of finite Galois extensions of K contained in $\overline{K}$.

Lemma 4.2. *Let D be a CDA over K and L/K a field extension contained in D. Then the following are equivalent:*

(i) *L is a special subfield of D.*
(ii) *D splits over L.*
(iii) *L is a maximal subfield of D.*

Proof. First we prove (i) $\iff$ (ii). We already know (i) $\implies$ (ii). If (ii) holds, then $D \otimes_K L$ is isomorphic to a matrix algebra over L. On the other hand, the centralizer theorem shows that

$$D \otimes_K L \simeq M_n(C_D(L))$$

and that $C_D(L)$ is a CDA over L. Therefore we get $L = C_D(L)$, that is, (i) holds.

Next we prove (i) $\iff$ (iii). Suppose that (i) holds. If E/L is a field extension contained in D, then we have $E \subset C_D(L) = L$, so (iii) holds. Conversely, if (i) does not hold, then we can form a non-trivial field extension $L[x]/L$ contained D by choosing $x \in C_D(L) \setminus L$. □

Lemma 4.3. *Let D be a CDA over K. If $D \neq K$, then there is a non-trivial separable extension L/K contained in D.*

Proof. It suffices to show that if $K[x]/K$ is purely inseparable for all $x \in D \setminus K$ then $D = K$. We may assume that K is an infinite field of characteristic $p > 0$. Our hypothesis implies $D^{p^r} \subset K$ for some $r \geq 1$.

Now let X_D be the ring scheme over K representing the functor

$$\mathbf{Alg}_K \to \mathbf{Ring};\ R \mapsto R \otimes_K D$$

and Z its closed subscheme representing $R \mapsto R \otimes_K K$. Note that the underlying K-scheme of X_D is isomorphic to $\mathbb{A}_K^{n^2}$ where $[D : K] = n^2$. Since K is infinite, K-rational points are dense in X_D, so the morphism

$$(-)^{p^r} : X_D \to X_D$$

factors through Z. Considering $\overline{K}$-valued points we get $M_n(\overline{K})^{p^r} \subset \overline{K}$ and hence $n = 1$. □

Proof of Theorem 4.1. Let D be a CDA over K and set $[D : K] = n^2$. It suffices to show that there is a separable extension L/K contained in D such that $[L : K] = n$. We inductively construct pairs (K_i, D_i) where K_i/K is a separable extension and D_i is a CDA over K_i such that $K \subset K_i \subset D_i \subset D$, $[D_i : K][K_i : K] = n^2$. Set $D_1 = D$ and $K_1 = K$. Suppose that (D_i, K_i) is defined. If $[K_i : K] = n$ then we are done. Otherwise, there is a non-trivial separable extension L/K_i contained in D_i by Lemma 4.3. Then set $K_{i+1} = L$ and $D_{i+1} = C_D(K_{i+1})$. It is clear that this process terminates and hence we eventually get a desired subfield. □

5. Brauer groups and Galois cohomology

Let K be a field and L/K a finite Galois extension with Galois group G. For any finite dimensional K-algebra A, its base change $L \otimes_K A$ has a semilinear action of G given by $\sigma(x \otimes a) = \sigma(x) \otimes a$.

Theorem 5.1 (Galois descent)**.** *$A \mapsto L \otimes_K A$ gives an equivalence of categories*

$$\begin{pmatrix} \text{Finite dimensional} \\ K\text{-algebras} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{Finite dimensional } L\text{-algebras} \\ \text{with a semilinear } G\text{-action} \end{pmatrix}.$$

Corollary 5.2. *$A \mapsto L \otimes_K A$ gives an equivalence of categories*

$$\begin{pmatrix} n^2\text{-dimensional CSAs over } K \\ \text{which split over } L \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} L\text{-algebras isomorphic to } M_n(L) \\ \text{with a semilinear } G\text{-action} \end{pmatrix}.$$

Let $B_n(L/K)$ denote the set of isomorphism classes of objects in the category on the left. We will describe this set by classifying semilinear G-actions on $M_n(L)$.

First we note that the automorphism group of the L-algebra $M_n(L)$ is $\mathrm{PGL}_n(L)$ (acting by $g \cdot x = gxg^{-1}$) by the Skolem-Noether theorem. Let $m_\sigma \colon M_n(L) \to M_n(L)$ be the K-linear map given by applying σ to each matrix components. Given a semilinear action of G on $M_n(L)$, we can write $\sigma \cdot x = m_\sigma(g_\sigma x g_\sigma^{-1})$ for a unique $g_\sigma \in \mathrm{PGL}_n(L)$. The map

$$\varphi \colon G \to \mathrm{PGL}_n(L); \quad \sigma \mapsto g_{\sigma^{-1}}^{-1}$$

gives a 1-cocycle:

$$\varphi(\sigma\tau) = \varphi(\sigma) \cdot \sigma\varphi(\tau).$$

Conversely, any 1-cocycle $\varphi \colon G \to \mathrm{PGL}_n(L)$ defines a semilinear action of G on $M_n(L)$ by

$$\sigma \cdot x = m_\sigma(\varphi(\sigma^{-1})^{-1} x \varphi(\sigma^{-1})).$$

One can check that two cocycles give isomorphic actions if and only if they are cohomologous. We have proved the following theorem.

Theorem 5.3. *There is a canonical isomorphism $\rho_n \colon B_n(L/K) \xrightarrow{\sim} H^1(G, \mathrm{PGL}_n(L))$ of pointed sets.*

Now we use the short exact sequence

$$1 \to L^\times \to \mathrm{GL}_n(L) \to \mathrm{PGL}_n(L) \to 1$$

of non-abelian G-modules. Since $L^\times \subset C(\mathrm{GL}_n(L))$, we get an exact sequence

$$H^1(G, \mathrm{GL}_n(L)) \to H^1(G, \mathrm{PGL}_n(L)) \xrightarrow{\delta} H^2(G, L^\times).$$

We have $H^1(G, \mathrm{GL}_n(L))$ by Hilbert's theorem 90 and hence we get a monomorphism

$$\iota_n \colon B_n(L/K) \xrightarrow[\sim]{\rho_n} H^1(G, \mathrm{PGL}_n(L)) \overset{\delta}{\hookrightarrow} H^2(G, L^\times).$$

Lemma 5.4. *Let A and B be CSAs over K which split over L. Set $[A : K] = m^2$ and $[B : K] = n^2$. Then $\iota_{mn}([A \otimes_K B]) = \iota_m([A]) + \iota_n([B])$.*

Proof. One can easily check that $\rho_{mn}([A \otimes_K B]) = \rho_m([A]) \otimes \rho_n([B])$, where $\otimes$ on the right hand side is induced by the Kronecker product $\otimes \colon \mathrm{PGL}_m(L) \times \mathrm{PGL}_n(L) \to \mathrm{PGL}_{mn}(L)$. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & L^\times \times L^\times & \longrightarrow & \mathrm{GL}_m(L) \times \mathrm{GL}_n(L) & \longrightarrow & \mathrm{PGL}_m(L) \times \mathrm{PGL}_n(L) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes} & & \\
1 & \longrightarrow & L^\times & \longrightarrow & \mathrm{GL}_{mn}(L) & \longrightarrow & \mathrm{PGL}_{mn}(L) & \longrightarrow & 1.
\end{array}$$

This yields the commutative diagram

$$\begin{array}{ccc}
H^1(G, \mathrm{PGL}_m(L)) \times H^1(G, \mathrm{PGL}_n(L)) & \overset{\delta \times \delta}{\hookrightarrow} & H^2(G, L^\times) \times H^2(G, L^\times) \\
\downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle +} \\
H^1(G, \mathrm{PGL}_{mn}(L)) & \overset{\delta}{\hookrightarrow} & H^2(G, L^\times)
\end{array}$$

and hence the result. □

This lemma implies that the following diagram is commutative, where the vertical map is given by $[A] \mapsto [M_n(A)]$:

$$\begin{array}{ccc}
B_m(L/K) & \overset{\iota_m}{\hookrightarrow} & H^2(G, L^\times) \\
\downarrow & \nearrow{\scriptstyle \iota_{mn}} & \\
B_{mn}(L/K). & &
\end{array}$$

Since $\mathrm{Br}(L/K) \simeq \bigcup_n B_n(L/K)$, we get a monomorphism $\iota \colon \mathrm{Br}(L/K) \to H^2(G, L^\times)$. Moreover, Lemma 5.4 implies that ι is a group homomorphism.

Theorem 5.5. *$\iota \colon \mathrm{Br}(L/K) \to H^2(G, L^\times)$ is an isomorphism.*

Proof. It suffices to show that any element of $H^2(G, L^\times)$ comes from $H^1(G, \mathrm{PGL}_n(L))$ for some $n \geq 1$. Let $\psi \colon G \times G \to L^\times$ be a 2-cocycle. Let V be a L-vector space with basis $\{e_\sigma \mid \sigma \in G\}$ and define an L-linear map $\varphi(\sigma) \colon V \to V;\ e_\tau \mapsto \psi(\sigma, \tau) e_{\sigma\tau}$. Then φ gives a 1-cocycle $G \to \mathrm{PGL}(V)$:

$$\varphi(\sigma\tau) = \varphi(\sigma) \cdot \sigma\varphi(\tau).$$

One can easily check that $\delta[\varphi] = [\psi]$. □

Example 5.6. We have $\mathrm{Br}(\mathbb{R}) = \mathrm{Br}(\mathbb{C}/\mathbb{R}) \simeq H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$. Since $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic, this is isomorphic to $H^0_T(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \simeq \mathbb{R}^\times / \mathrm{Nm}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \simeq \mathbb{Z}/2$. Actually, one can check that $\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$.

Example 5.7. For a prime power q and $n \geq 1$, we have $\mathrm{Br}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq H^2(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \mathbb{F}_{q^n}^\times)$. Since $\mathbb{F}_{q^n}$ is finite, we have $h(\mathbb{F}_{q^n}^\times) = 1$ and hence $\#\,\mathrm{Br}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \# H^1(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \mathbb{F}_{q^n}^\times) = 1$ by Hilbert's theorem 90. Therefore $\mathrm{Br}(\mathbb{F}_q) = 0$.

6. Brauer group of local fields

Let K be a non-archimedean local field with valuation v_K. Let $\mathcal{O}_K$ be the ring of integers of K, $\mathfrak{m}_K$ its maximal ideal and $k = \mathcal{O}_K/\mathfrak{m}_K$. We prove the following theorem.

Theorem 6.1. $\mathrm{Br}(K) = \mathrm{Br}(K^{\mathrm{ur}}/K)$, *i.e. any CSA over K splits over some finite unramified extension of K.*

Let D be a CDA over K and set $[D : K] = n^2$. First we define a function $v_D \colon D \to \mathbb{Q} \cup \{\infty\}$ by

$$v_D(x) = \frac{1}{n^2} v_K(\det(D \xrightarrow{x\cdot(-)} D)).$$

Lemma 6.2. *The following hold:*

(i) *If L/K is a field extension contained in D, then $v_D|_L = v_L$ where v_L is the unique discrete valuation extending v_K.*
(ii) $v_D(xy) = v_D(x) + v_D(y)$.
(iii) $v_D(x + y) \geq \min\{v_D(x), v_D(y)\}$.

Proof.

(i) Set $d = [L : K]$. Then D can be regarded an L-vector space of dimension n^2/d by left multiplication, and hence for any $x \in L$ we have

$$v_D(x) = \frac{1}{n^2} v_K(\det(L \xrightarrow{x\cdot(-)} L)^{n^2/d}) = \frac{1}{d} v_K(\mathrm{Nm}_{L/K}(x)) = v_L(x).$$

(ii) This is obvious from the definition.
(iii) By (iii) it suffices to prove $v_D(z) \geq 0 \implies v_D(1 + z) \geq 0$ for $z \in D^\times$. This can be seen by applying (i) to $K[z]$.

□

We define $\mathcal{O}_D := \{x \in D \mid v_D(x) \geq 0\}$ and $\mathfrak{m}_D := \{x \in D \mid v_D(x) > 0\}$. Then $\mathcal{O}_D$ is a local $\mathcal{O}_K$-algebra with maximal two-sided ideal $\mathfrak{m}_D$, and $F := \mathcal{O}_D/\mathfrak{m}_D$ is a division algebra over k ($[F : K] < \infty$ can be seen by lifting a basis). By Example 5.7, F is a finite extension field of k. We define e_D to be the positive integer satisfying $v_D(D^\times) = \frac{1}{e}\mathbb{Z}$.

Lemma 6.3. *If $D \neq K$, then $[\mathcal{O}_D/\mathfrak{m}_D : k] > 1$.*

Proof. Suppose that $[\mathcal{O}_D/\mathfrak{m}_D : k] = 1$. Choose a uniformizer $\pi \in \mathcal{O}_K$, an element $\Pi \in \mathcal{O}_D$ with $v_D(\Pi) = 1/e$ and a system of representatives $S \subset \mathcal{O}_K$ of k. Then $\Pi^j \pi^i S$ gives a system of representatives of $\mathfrak{m}_D^{ei+j}/\mathfrak{m}_D^{ei+j+1}$ where $i \in \mathbb{Z}_{\geq 0}$ and $0 \leq j < e$. Therefore any element of $\mathcal{O}_D$ can be written uniquely as

$$\sum_{j=0}^{e-1} \Pi^j \left(\sum_{i=0}^{\infty} \pi^i s_{ij} \right) \quad (s_{ij} \in S).$$

It follows that $K[\Pi] = D$, which is a contradiction. □

Proof of Theorem 6.1. Suppose that $D \neq K$. By Lemma 6.3, we can choose an element $\overline{a} \in F \setminus k$ which is separable over k. Let $a \in \mathcal{O}_D$ be a lift of $\overline{a}$. Then $K[a]/K$ is an extension with a non-trivial residue field extension, so there is a non-trivial unramified subextension L/K. An argument as in the proof of Theorem 4.1 shows that there is a special subfield of D which is unramified over K. □